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On the matter of the Dijkgraaf-Vafa conjecture

Per Kraus and Masaki Shigemori

Department of Physics and Astronomy
UCLA, Los Angeles, CA 90095-1547, US
E-mail: pkraus@physics.ucla.edu, shige@physics.ucla.edu

Abstract: With the aim of extending the gauge theory-matrix model connection to more general matter representations, we prove that for various two-index tensors of the classical gauge groups, the perturbative contributions to the glueball superpotential reduce to matrix integrals. Contributing diagrams consist of certain combinations of spheres, disks, and projective planes, which we evaluate to four and five loop order. In the case of $\mathrm{Sp}(N)$ with antisymmetric matter, independent results are obtained by computing the nonperturbative superpotential for $N = 4, 6$ and 8. Comparison with the Dijkgraaf-Vafa approach reveals agreement up to $N/2$ loops in matrix model perturbation theory, with disagreement setting in at $h = N/2 + 1$ loops, h being the dual Coxeter number. At this order, the glueball superfield S begins to obey nontrivial relations due to its underlying structure as a product of fermionic superfields. We therefore find a relatively simple example of an $\mathcal{N} = 1$ gauge theory admitting a large-N expansion, whose dynamically generated superpotential differs from the one obtained in the matrix model approach.

Keywords: Nonperturbative Effects, Confinement, Superspaces, Matrix Models.

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Contents

1. Introduction

The methods of Dijkgraaf and Vafa [1, 2, 3] represent a potentially powerful approach to obtaining nonperturbative results in a wide class of supersymmetric gauge theories. Their original conjecture consists of two parts. First, that holomorphic physics is captured by an effective superpotential for a glueball superfield, with nonperturbative effects included via the Veneziano-Yankielowicz superpotential [4]. Second, that the Feynman diagrams contributing to the perturbative part of the glueball superpotential reduce to matrix model diagrams.

The second part of the conjecture has been proven for a few choices of matter fields and gauge groups, namely U(N) with adjoint [5, 6] and fundamental [7] matter, and SO / Sp(N) with adjoint matter [8, 9, 10]. Combining this with the first part of the conjecture has then been shown to reproduce known gauge theory results. Some examples of "exotic" tree-level superpotentials have also been considered successfully, such as multiple trace [11] and baryonic [12] interactions.

One naturally wonders how far this can be pushed. Generic $\mathcal{N} = 1$ theories possess intricate dynamically generated superpotentials which are difficult or (nearly) impossible to obtain by traditional means, and so a systematic method for computing them would be most welcome. The promise of the DV approach is that these perhaps can be obtained to any desired order by evaluating matrix integrals. With this in mind, we will demonstrate the reduction to matrix integrals for some new matter representations. We will then find some impressive agreements, as well as obstacles, when comparing to known gauge theory results

In particular, it is straightforward to generalize the results of [5, 8] to more general two-index tensors of $\mathrm{U}(N)$ and $\mathrm{SO}\,/\,\mathrm{Sp}(N)$, with or without tracelessness conditions imposed. The relevant $0+0$ dimensional Feynman diagrams which one needs to compute consist of various spheres, disks and projective planes, and disconnected sums of these. We evaluate these to five-loop order.

For comparison with gauge theory we focus on the particular case of $\mathrm{Sp}(N)$[1] with an antisymmetric tensor chiral superfield. The dynamically generated superpotentials for such theories are highly nontrivial, and cannot be obtained via the "integrating in" approach of [13]. Furthermore, the results display no simple pattern in N. Nevertheless, a method is known for computing these superpotentials on a case-by-case basis [14]. Results for Sp(4) and Sp(6) were obtained in [14], and here we extend this to Sp(8) as well (partial results for Sp(8) appear in [14]). We believe that these examples illustrate the main features of generic $\mathcal{N} = 1$ superpotentials, and so are a good testing ground for the DV approach.

For our $\mathrm{Sp}(N)$ examples, we will demonstrate agreement between our gauge theory superpotentials and the DV approach up to $N/2$ loops in perturbation theory, with a disagreement setting in at $N/2+1$ loops. In terms of the glueball superpotential, we thus find a disagreement at order S^h, where $h = N/2+1$ is the dual Coxeter number of $\mathrm{Sp}(N)$.

Given that discrepancies occur, it is perhaps not surprising that they arise at order S^h, for it is at this order that S begins to obey relations due to its being a product of two fermionic superfields [6, 15]. Furthermore, at this order contributions to the effective action for W_α of the schematic form $\mathrm{Tr}\ (W_\alpha)^{2h}$ can be reexpressed in terms of lower traces, including S^h. Unfortunately, it is not clear how to ascertain these relations *a priori*, since they receive corrections from nonperturbative effects (see [15] for a recent discussion). These complications do not arise for theories with purely adjoint matter, since unlike in our examples, the gauge theory results are known to have a simple pattern in N, and so N can be formally taken to infinity to avoid having to deal with any relations involving the S's. There are also a number of other potential subtleties involved, as we will discuss in section 5. In any case, it seems that additional input is required to make progress at h loops and beyond.

The remainder of this paper is organized as follows. In section 2 we isolate the field theory diagrams that contribute to the glueball superpotential, derive the reduction of these diagrams to those of a matrix model, and discuss their computation. These results are used in section 3 to derive effective superpotentials for $\mathrm{Sp}(N)$ with matter in the antisymmetric

[1] Our convention for $\mathrm{Sp}(N)$ is such that N is an even integer, and $\mathrm{Sp}(2) \approx \mathrm{SU}(2)$.

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tensor representation. In section 4 we state the corresponding results derived from a nonperturbative superpotential for these theories. Comparison reveals a discrepancy, which we discuss in section 5. Appendix A gives more details on diagram calculations; appendix B collects results from matrix model perturbation theory; and appendix C concerns the computation of dynamically generated superpotentials for the $\mathrm{Sp}(N)$ theories.

2. Reduction to matrix model

In this section we will extend the results of [5, 8] to include the following matter representations:

- $\mathrm{U}(N)$ adjoint.
- $\mathrm{SU}(N)$ adjoint.
- $\mathrm{SO}(N)$ antisymmetric tensor
- $\mathrm{SO}(N)$ symmetric tensor, traceless or traceful.
- $\mathrm{Sp}(N)$ symmetric tensor.
- $\mathrm{Sp}(N)$ antisymmetric tensor, traceless or traceful.

We will use Φ_{ij} to denote the matter superfield. In the case of $\mathrm{Sp}(N)$, Φ_{ij} is defined as

$$\Phi = \begin{cases} SJ & S_{ij}\text{: symmetric tensor}\,, \\ AJ & A_{ij}\text{: antisymmetric tensor}\,. \end{cases} \tag{2.1}$$

Here J is the invariant antisymmetric tensor of $\mathrm{Sp}(N)$, namely

$$J_{ij} = \begin{pmatrix} 0 & \mathbf{1}_{N/2} \\ -\mathbf{1}_{N/2} & 0 \end{pmatrix}. \tag{2.2}$$

The tracelessness of the Sp antisymmetric tensor is defined with respect to this J, i.e., by $\mathrm{Tr}[AJ] = 0$.

The fact that allows us to treat the above cases in parallel to those considered in [5, 8] is that gauge transformations act by commutation, $\delta_\Lambda \Phi \sim [\Lambda, \Phi]$. A separate analysis is needed for, say, $\mathrm{U}(N)$ with (anti)symmetric matter (see [16] for some work on such cases).

2.1 Basic setup

Following [5], we consider a supersymmetric gauge theory with chiral superfield Φ and field strength $\mathcal{W}^\alpha$. Treating $\mathcal{W}^\alpha$ as a fixed background, we integrate out Φ to all orders in perturbation theory. We are interested in the part of the effective action which takes the form of a superpotential for the glueball superfield $S = \frac{1}{32\pi^2}\,\mathrm{Tr}[\mathcal{W}^\alpha \mathcal{W}_\alpha]$. In [5], using the superspace formalism, it was shown that this can be obtained from a simple action involving only chiral superfields:

$$S(\Phi) = \int d^4p\, d^2\pi \left[\frac{1}{2}\Phi\left(p^2 + \mathcal{W}^\alpha \pi_\alpha\right)\Phi + W_{\mathrm{tree}}(\Phi)\right]. \tag{2.3}$$

We choose the tree level superpotential to be

$$W_{\text{tree}} = \frac{m}{2} \operatorname{Tr}\left[\Phi^2\right] + \text{interactions}\,, \tag{2.4}$$

where the interactions are single trace terms, and include the mass in the propagator:

$$\frac{1}{p^2 + m + \mathcal{W}^\alpha \pi_\alpha}\,. \tag{2.5}$$

Actually, we have to be a little more precise here. Displaying all indices, we can write the quadratic action as

$$\frac{1}{2}\int d^4p\, d^2\pi\; \Phi_{ji} G^{-1}_{ijkl} \Phi_{kl} \tag{2.6}$$

with

$$G^{-1}_{ijkl} = \left[\left(p^2 + m\right)\delta_{im}\delta_{jn} + \left(\mathcal{W}^\alpha\right)_{ijmn}\pi_\alpha\right] P_{mnkl}\,. \tag{2.7}$$

Here the P's are projection operators appropriate for the gauge group and matter representation under consideration:

$$P_{ijkl} = \begin{cases} \delta_{ik}\delta_{jl} & \text{U}(N) \text{ adjoint} \\ \delta_{ik}\delta_{jl} - \frac{1}{N}\delta_{ij}\delta_{kl} & \text{SU}(N) \text{ adjoint} \\ \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) & \text{SO}(N) \text{ antisymmetric} \\ \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) & \text{SO}(N) \text{ traceful symmetric} \\ \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{N}\delta_{ij}\delta_{kl}) & \text{SO}(N) \text{ traceless symmetric} \\ \frac{1}{2}(\delta_{ik}\delta_{jl} - J_{il}J_{jk}) & \text{Sp}(N) \text{ symmetric} \\ \frac{1}{2}(\delta_{ik}\delta_{jl} + J_{il}J_{jk}) & \text{Sp}(N) \text{ traceful antisymmetric} \\ \frac{1}{2}\left(\delta_{ik}\delta_{jl} + J_{il}J_{jk} - \frac{2}{N}\delta_{ij}\delta_{kl}\right) & \text{Sp}(N) \text{ traceless antisymmetric}\,. \end{cases} \tag{2.8}$$

The propagator is then given by the inverse of G^{-1} in the subspace spanned by P:

$$\langle \Phi_{ji}\Phi_{kl}\rangle = \left[\frac{P}{p^2 + m + \mathcal{W}^\alpha\pi_\alpha}\right]_{ijkl} = \left[\int_0^\infty ds\, e^{-s(p^2+m+\mathcal{W}^\alpha\pi_\alpha)} P\right]_{ijkl}\,. \tag{2.9}$$

Our rule for multiplying four-index objects is $(AB)_{ijkl} = \sum_{mn} A_{ijmn}B_{mnkl}$. The fact that gauge transformations act by commutation means that we can write

$$(\mathcal{W}^\alpha)_{ijkl} = (\mathcal{W}^\alpha)_{ik}\delta_{jl} - (\mathcal{W}^\alpha)_{lj}\delta_{ik} \tag{2.10}$$

where on the right hand side $(\mathcal{W}^\alpha)_{ij}$ are field strengths in the defining representation of the gauge group.

2.2 Diagrammatics

The presence of the three sorts of terms in the projection operators (2.8) means that in double line notation we have three types of propagators, displayed in figure 1.

Note in particular the disconnected propagator, which allows us to draw Feynman diagrams which have disconnected components in index space (All diagrams are connected in momentum space since we are computing the free energy). A typical diagram involving cubic interactions is shown in figure 2.

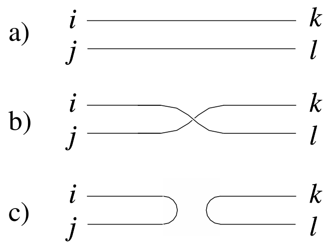

Figure 1: Propagators. *a*) untwisted; *b*) twisted; *c*) disconnected.

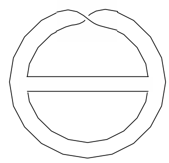

Figure 2: Typical diagram.

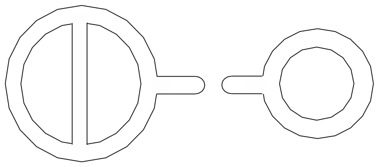

Figure 3: Contributing diagram.

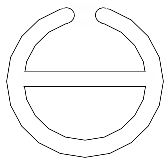

Figure 4: Non-contributing diagram.

Since we are computing the superpotential for S, we include either zero or two insertions of $\mathcal{W}^\alpha$ on each index loop.[2] We will now prove that the diagrams which contribute are those consisting of some number of sphere, disk, and projective plane components. Furthermore, the total number of disconnected components must be one greater than the number of disconnected propagators. Figure 3 is an example of a contributing diagram, while figure 4 is a diagram which does not contribute.

The proof is is similar to that given in [5, 8], so we mainly focus on the effect of the new disconnected propagator. In double line notation we associate each Feynman diagram to a two-dimensional surface. Let F by the number of faces (index loops); P be the number of edges; and V be the number of vertices. The Feynman diagram also has some number L of momentum loops. Euler's theorem tells us that

$$F = P - V + \chi \tag{2.11}$$

where $\chi_{S^2} = 2$, $\chi_{D^2} = \chi_{RP^2} = 1$. We also have the relation

$$F = L - 1 + \chi\,. \tag{2.12}$$

In a diagram with L loops we need to bring down L powers of S to saturate the fermion integrals, and we allow at most one S per index loop. Therefore, for a graph to be nonvanishing we need $F \geq L$. Graphs on S^2 with no disconnected propagators have $F = L + 1$, and those on RP^2 have $F = L$.

To proceed we will make use of the following operation. Considering some diagram D that includes some number of disconnected propagators. To each D we associate a diagram $\tilde{D}$, obtained by replacing each disconnected propagator of D by an untwisted propagator.

[2] Note that we are explicitly *not* including the contributions coming from more than two $\mathcal{W}^\alpha$'s on an index loop, even if for a particular N these can be expressed in terms of S's. We will come back to this point in section 5.

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Each $\tilde{D}$ diagram thus consists of a single connected component. D and $\tilde{D}$ have the same values of L and V, but can have different values of F, P, and χ. We use $\tilde{F}$, $\tilde{P}$, and $\tilde{\chi}$ to denote the number of faces, edges, and the Euler number of $\tilde{D}$. To see which diagrams can contribute we consider various cases.

Case 1: D has no disconnected propagators, so $\tilde{D} = D$. This case reduces to that of [5, 8], and so we know that only S^2 and RP^2 graphs contribute (since no D^2 graphs can arise without disconnected propagators, these are the only graphs for which $F \geq L$.)

The remaining cases to consider are those for which we have at least one disconnected propagator.

Case 2: $\chi = \tilde{\chi} \leq 1$. In this case $\tilde{F} \leq L$, from (2.12). Each time we take a disconnected propagator and replace it by an untwisted propagator we are increasing P by 1 but keeping L unchanged. Therefore, from (2.11), this operation increases F by 1. So we see that in this case $F < L$. This means that the diagram D does not contribute.

Case 3: $\chi = \tilde{\chi} = 2$. $\tilde{D}$ has $\tilde{F} = L + 1$. In this case, if D has a single disconnected propagator, we will have $F = L$, and so the diagram might seem to contribute. But we will now show that the fermion determinant vanishes for such diagrams.

We follow the conventions of [8], where the reader is referred for more details. The fermion contribution is proportional to $[\det N(s)]^2$ where

$$N(s)_{ma} = \sum_i s_i K^T_{mi} L_{ia} \,. \tag{2.13}$$

Here, i labels propagators; m labels "active" index loops on which we insert an S; and a labels momentum loops. In the present case, since $F = L$, all index loops are active and so N is a square matrix. To show that the determinant vanishes, we will show that the rectangular matrix

$$s_i K_{im} \tag{2.14}$$

has a nontrivial kernel.

Recall the definition of K_{im}. For each oriented propagator labelled by i, the mth index loop can do one of three things: 1) coincide and be parallel, giving $K_{im} = 1$; 2) coincide and be anti-parallel, giving $K_{im} = -1$; 3) not coincide, giving $K_{im} = 0$. Consider K_{im} acting on the vector b_m whose components are all equal to 1. It should be clear that

$$\sum_m K_{im} b_m = 1 - 1 = 0 \,. \tag{2.15}$$

The intuitive way to think about this is that b_m are the index loop momenta and $\sum_m K_{im} b_m$ are the propagator momenta. By setting all index loop momenta equal, one makes all propagator momenta vanish, and this corresponds to an element of the kernel of (2.14). This finally implies $\det N(s) = 0$, which is what we wanted to show.

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Case 4: $\chi \neq \tilde{\chi}$**.** This can only happen when D has two or more disconnected components. In this case, $\chi > \tilde{\chi}$ and so (2.11) still allows $F \geq L$ even when $P < \tilde{P}$.

In order to have a nonvanishing fermion integral, each component of D must have $F \geq L$, so each component must be an S^2, a D^2, or an RP^2. Suppose D has N_{S^2} S^2 components, N_{D^2} D^2 components, and N_{RP^2} RP^2 components, so that

$$\chi = 2N_{S^2} + N_{RP^2} + N_{D^2} . \tag{2.16}$$

Next consider the relation between P and $\tilde{P}$. The number of disconnected propagators must be at least the number of disconnected components of D minus one, so

$$P = \tilde{P} - (N_{S^2} + N_{RP^2} + N_{D^2} - 1) - a = \tilde{P} + 1 + N_{S^2} - \chi - a \tag{2.17}$$

where a is a nonnegative integer. Now use

$$F = P - V + \chi = \tilde{P} + 1 + N_{S^2} - V - a . \tag{2.18}$$

$\tilde{D}$ satisfies

$$\tilde{P} + 1 - V = L \tag{2.19}$$

so we get

$$F = L + N_{S^2} - a . \tag{2.20}$$

Now, in order to have a nonzero fermion determinant we need to have at least one inactive index loop (no $\mathcal{W}^a$ insertions) per S^2 component after choosing L active index loops. In other words, a nonvanishing fermion determinant requires

$$F \geq L + N_{S^2} . \tag{2.21}$$

Putting these two conditions together, we clearly need $a = 0$. This says that the number of disconnected propagators in D must be precisely equal to the number of disconnected components of D minus one.

Summary: Diagrams which contribute to the glueball superpotential have any number of disconnected S^2, D^2, and RP^2 components. The number of disconnected propagators must be one less than the number of disconnected components.[3]

2.3 Computation of diagrams

Now that we have isolated the class of diagrams which contribute to the glueball superpotential, we turn to their computation. This turns out to be a simple extension of what is already known. In particular, the contribution from a general disconnected diagram is simply equal to an overall combinatorial factor times the product of the contributions of the individual components. This follows from the fact that, for the diagrams we are considering, the disconnected propagators carry vanishing momentum, so the diagrams are actually disconnected in both momentum space and index space.

[3]It is easy to convince oneself that disconnected diagrams will therefore never contribute in theories with only even powers in the tree level superpotential, thus giving the same glueball superpotential in the traceful and traceless cases.

Next, we observe that the stubs from the disconnected propagators can be neglected in the computation; it is easily checked that the sum over $\mathcal{W}^a$ insertions on the stubs gives zero due to the minus sign in (2.10).

So we just need rules for treating each component individually, and then we multiply the contributions together to get the total diagram. The rules consist of relating the gauge theory contribution to a corresponding matrix contribution. The cases of interest are:

S^2 **components:** From the work of [5], we know that if $N^{L+1}F^{(L)}_{S^2}(g_k)$ is the contribution in the matrix model diagram from an L loop S^2 graph, then the contribution in the gauge theory diagram is

$$W^{(L)}_{S^2}(S, g_k) = (L+1)NS^L F^{(L)}_{S^2}(g_k) \tag{2.22}$$

The prefactor $(L+1)N$ comes from the choice of, and trace over, a single inactive index loop.

RP^2 **components:** From the work of [8], we know that if $NF^{(L)}_{RP^2}(g_k)$ is the contribution in the matrix model diagram from an L loop RP^2 graph, then the contribution in the gauge theory diagram is

$$W^{(L)}_{RP^2}(S, g_k) = \pm 4S^L F^{(L)}_{RP^2}(g_k) \tag{2.23}$$

The prefactor of ± 4 comes from the fermion determinant, and is equal to $+4(-4)$ for symmetric(antisymmetric) tensors.

D^2 **components:** These have $L = 0$ and hence no $\mathcal{W}^a$ insertions. So if the contribution to the matrix model is $NF^{(L)}_{D^2}(g_K)$ then

$$W^{(L)}_{D^2}(g_k) = NF^{(L)}_{D^2}(g_K)\,. \tag{2.24}$$

With the above rules in hand, it is a simple matter to convert a given matrix model Feynman diagram into a contribution to the glueball superpotential. The example given in appendix A should help to clarify this. We should emphasize that the above procedure must by done diagram by diagram — there is no obvious way to directly relate the entire glueball superpotential to the matrix model free energy; the situation is similar to [11] in this respect.

3. Results from matrix integrals

The considerations thus far apply to any single trace, polynomial, tree level superpotential. We now restrict attention to cubic interactions,

$$W_{\rm tree} = \frac{m}{2}\,{\rm Tr}\ \Phi^2 + \frac{g}{3}\,{\rm Tr}\ \Phi^3\,. \tag{3.1}$$

(which are of course trivial in the case of SO / Sp with adjoint matter.) In appendix B we collect our matrix model results for the various matter representations. In this section we focus on two particular cases, which will be compared to gauge theory results in the next section.

3.1 Sp(N) with traceful antisymmetric matter

The perturbative part of the glueball superpotential for Sp(N) with traceful antisymmetric matter is

$$W^{\text{pert}}_{\text{traceful}}(S,\alpha) = (-N+3)\,\alpha S^2 + \left(-\frac{16}{3}N + \frac{59}{3}\right)\alpha^2 S^3 + \left(-\frac{140}{3}N + 197\right)\alpha^3 S^4 + \\ + \left(-512N + \frac{4775}{2}\right)\alpha^4 S^5 + \cdots \tag{3.2}$$

with

$$\alpha \equiv \frac{g^2}{2m^3}\,. \tag{3.3}$$

In terms of diagrams, (3.2) represents the contribution from $2,3,4$ and 5 loops. According to the DV conjecture, the full glueball superpotential is then $W^{\text{eff}} = W^{VY} + W^{\text{pert}}$, where W^{VY} is the Veneziano-Yankielowicz superpotential:

$$W^{VY} = \left(\frac{N}{2}+1\right) S\left[1 - \log\left(\frac{S}{\Lambda^3}\right)\right]. \tag{3.4}$$

We are now instructed to extremize W^{eff} with respect to S and substitute back in. We call the result W^{DV}. Working in a power series in g, we obtain

$$W^{\text{DV}}_{\text{traceful}}(\Lambda,m,g) = \left(\frac{N}{2}+1\right)\Lambda^3\left[1 - \frac{2(N-3)}{N+2}\Lambda^3\alpha - \frac{2(4N^2+45N-226)}{3(N+2)^2}\Lambda^6\alpha^2 - \right. \\ - \frac{2(12N^3+293N^2+368N-8340)}{3(N+2)^3}\Lambda^9\alpha^3 - \\ - \frac{96N^4+3803N^3+25868N^2-85092N-744768}{3(N+2)^4}\times \\ \left. \times\,\Lambda^{12}\alpha^4 - \cdots\right]. \tag{3.5}$$

For $N = 4,6,8$, this yields

$$\begin{aligned} W^{\text{DV},\text{Sp}(4)}_{\text{traceful}}(\Lambda,\alpha) &= 3\Lambda^3 - \Lambda^6\alpha - \Lambda^9\alpha^2 + \frac{353}{27}\Lambda^{12}\alpha^3 + \frac{25205}{81}\Lambda^{15}\alpha^4 + \cdots \\ W^{\text{DV},\text{Sp}(6)}_{\text{traceful}}(\Lambda,\alpha) &= 4\Lambda^3 - 3\Lambda^6\alpha - \frac{47}{6}\Lambda^9\alpha^2 - \frac{73}{2}\Lambda^{12}\alpha^3 - \frac{6477}{32}\Lambda^{15}\alpha^4 - \cdots \\ W^{\text{DV},\text{Sp}(8)}_{\text{traceful}}(\Lambda,\alpha) &= 5\Lambda^3 - 5\Lambda^6\alpha - 13\Lambda^9\alpha^2 - 65\Lambda^{12}\alpha^3 - \frac{2142}{5}\Lambda^{15}\alpha^4 - \cdots\,. \end{aligned} \tag{3.6}$$

3.2 Sp(N) with traceless antisymmetric matter

Including the contribution from the disconnected propagator, the perturbative part of the glueball superpotential for Sp(N) with traceless antisymmetric matter is

$$W^{\text{pert}}_{\text{traceless}}(S,\alpha) = \left(-1+\frac{4}{N}\right)\alpha S^2 + \left(-\frac{1}{3} - \frac{8}{N} + \frac{160}{3N^2}\right)\alpha^2 S^3 \\ + \left(-\frac{1}{3} - \frac{12}{N} - \frac{256}{3N^2} + \frac{3584}{3N^3}\right)\alpha^3 S^4 + \cdots\,. \tag{3.7}$$

The presence of many disconnected diagrams makes this case more complicated than the traceful case, and we have correspondingly worked to one lower order than in (3.2).

Adding the Veneziano-Yankielowicz superpotential and integrating out the glueball superfield, we obtain

$$\begin{aligned} W^{\mathrm{DV}}_{\mathrm{traceless}}(\Lambda,m,g) = \left(\frac{N}{2}+1\right)\Lambda^3 \Bigg[1 &- \frac{2(N-4)}{N(N+2)}\Lambda^3\alpha - \\ &- \frac{2(N^3+14N^2-16N-512)}{3N^2(N+2)^2}\Lambda^6\alpha^2 - \\ &- \frac{2(N+8)^2(N^3+12N^2-52N-528)}{3N^3(N+2)^3}\Lambda^9\alpha^3 - \cdots \Bigg]. \end{aligned} \tag{3.8}$$

This yields

$$\begin{aligned} W^{\mathrm{DV,Sp(4)}}_{\mathrm{traceless}}(\Lambda,\alpha) &= 3\Lambda^3 + \Lambda^9\alpha^2 + 10\Lambda^{12}\alpha^3 + \cdots \\ W^{\mathrm{DV,Sp(6)}}_{\mathrm{traceless}}(\Lambda,\alpha) &= 4\Lambda^3 - \frac{1}{3}\Lambda^6\alpha - \frac{7}{54}\Lambda^9\alpha^2 + \frac{49}{54}\Lambda^{12}\alpha^3 + \cdots \\ W^{\mathrm{DV,Sp(8)}}_{\mathrm{traceless}}(\Lambda,\alpha) &= 5\Lambda^3 - \frac{1}{2}\Lambda^6\alpha - \frac{2}{5}\Lambda^9\alpha^2 - \frac{14}{25}\Lambda^{12}\alpha^3 - \cdots . \end{aligned} \tag{3.9}$$

4. Gauge theory example: $\mathrm{Sp}(N)$ with antisymmetric matter

Dynamically generated superpotentials can be determined for $\mathcal{N}=1$ theories with gauge group $\mathrm{Sp}(N)$ and a chiral superfield A_{ij} in the antisymmetric tensor representation. The general procedure was given in [14], and is reviewed in appendix C. Since these superpotentials cannot be obtained by the integrating in procedure of [13], they are more difficult to establish, and the results are correspondingly more involved, than more familiar examples. A separate computation is required for each N, and the results display no obvious pattern in N. $N=4$ is a simple special case (since $\mathrm{Sp}(4)\approx \mathrm{SO}(5)$ and $A_{ij}\approx$ vector); $N=6$ was worked out in [14], and in appendix C we extend this to Sp(8) (the second paper listed in [14] gives the result for Sp(8) with some additional fundamentals, which need to be integrated out for our purposes). In this section we state the results, and integrate out A_{ij} to obtain formulas that we can compare with the DV approach.

The moduli space of the classical theory is parameterized by the gauge invariant operators

$$O_n = \mathrm{Tr}[(AJ)^n]\,, \qquad n=1,2,\dots,\frac{N}{2} \tag{4.1}$$

the upper bound coming from the characteristic equation of the matrix AJ.

From the gauge theory point of view, it is natural to demand tracelessness, and this will be denoted by a tilde: $\mathrm{Tr}[\tilde{A}J]=0$,

$$\tilde{O}_n = \mathrm{Tr}\left[\left(\tilde{A}J\right)^n\right], \qquad n=2,\dots,\frac{N}{2}\,. \tag{4.2}$$

In comparing with the DV approach, we will consider both the traceless and traceful cases.

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4.1 Traceless case

The Sp(4) and Sp(6) dynamical superpotentials for these fields are [14]:

$$W_{\rm dyn}^{\rm Sp(4)} = \frac{2{\Lambda_0}^4}{\tilde{O}_2^{1/2}}, \tag{4.3}$$

$$W_{\rm dyn}^{\rm Sp(6)} = \frac{4{\Lambda_0}^5}{\tilde{O}_2[(\sqrt{R}+\sqrt{R+1})^{2/3}+(\sqrt{R}+\sqrt{R+1})^{-2/3}-1]}, \tag{4.4}$$

with $R = -12{\tilde{O}_3}^2/{\tilde{O}_2}^3$.

Also, as derived in appendix C, the Sp(8) superpotential is

$$\begin{aligned} W_{\rm dyn}^{\rm Sp(8)} &= \frac{6\sqrt{2}\,\Lambda_0^6}{\tilde{O}_2^{3/2}}[-36R_4+144b^2R_4+288cR_4+8R_3^2+192bcR_3+ \\ &\qquad +1152b^2c^2-36b^2-72c+9]^{-1}, \end{aligned} \tag{4.5}$$

where $R_3 \equiv \tilde{O}_3/{\tilde{O}_2}^{3/2}$, $R_4 \equiv \tilde{O}_4/{\tilde{O}_2}^2$, and b and c are determined by

$$\begin{aligned} 12R_4+16bR_3-192b^2c+24b^2+96c^2-3 &= 0, \\ 12bR_4+8b^2R_3+8R_3c-96bc^2+24bc-3b &= 0. \end{aligned} \tag{4.6}$$

We choose the root which gives $R_3 = 0$ as the solution of the F-flatness condition.

Now let us integrate out the antisymmetric matter. We add the tree level superpotential

$$W_{\rm tree} = \frac{m}{2}\tilde{O}_2 + \frac{g}{3}\tilde{O}_3 \tag{4.7}$$

to the dynamical part, solve the F-flatness equations, and substitute back in. We do this perturbatively in g, and obtain

$$\begin{aligned} W_{\rm traceless}^{\rm gt,Sp(4)} &= 3\Lambda^3, \\ W_{\rm traceless}^{\rm gt,Sp(6)} &= 4\Lambda^3 - \frac{1}{3}\Lambda^6\alpha - \frac{7}{54}\Lambda^9\alpha^2 - \frac{5}{54}\Lambda^{12}\alpha^3 - \frac{221}{2592}\Lambda^{15}\alpha^4 - \cdots, \\ W_{\rm traceless}^{\rm gt,Sp(8)} &= 5\Lambda^3 - \frac{1}{2}\Lambda^6\alpha - \frac{2}{5}\Lambda^9\alpha^2 - \frac{14}{25}\Lambda^{12}\alpha^3 - \Lambda^{15}\alpha^4 - \cdots, \end{aligned} \tag{4.8}$$

where α is defined in (3.3), and the low-energy scales are defined from the usual matching conditions as

$$(\Lambda^3)^{\frac{N}{2}+1} = \left(\frac{m}{2}\right)^{\frac{N}{2}-1}\Lambda_0^{N+4}. \tag{4.9}$$

4.2 Traceful case

For Sp(N) theory with a traceful antisymmetric tensor A_{ij}, we separate out the trace part as

$$A_{ij} = \tilde{A}_{ij} - \frac{1}{N}J_{ij}\phi, \qquad {\rm Tr}[\tilde{A}J] = 0, \qquad {\rm Tr}[AJ] = \phi. \tag{4.10}$$

$\tilde{O}_n$ are related to their traceful counterparts $O_n \equiv {\rm Tr}[(AJ)^n]$ by

$$O_2 = \tilde{O}_2 + \frac{\phi^2}{N}, \qquad O_3 = \tilde{O}_3 + \frac{3}{N}\tilde{O}_2\phi + \frac{1}{N^2}\phi^3. \tag{4.11}$$

The dynamical superpotential of this traceful theory is the same as the traceless theory, since ϕ has its own U(1)$_\phi$ charge and hence cannot enter in $W_{\rm dyn}$.

Integrating out $\tilde{A}_{ij}$ and ϕ in the presence of the tree level superpotential

$$W_{\rm tree} = \frac{m}{2} O_2 + \frac{g}{3} O_3 \,, \qquad (4.12)$$

we obtain

$$\begin{aligned} W^{\rm gt, Sp(4)}_{\rm traceful} &= 3\Lambda^3 - \Lambda^6\alpha - 2\Lambda^9\alpha^2 - \frac{187}{27}\Lambda^{12}\alpha^3 - \frac{2470}{27}\Lambda^{15}\alpha^4 - \cdots \,, \\ W^{\rm gt, Sp(6)}_{\rm traceful} &= 4\Lambda^3 - 3\Lambda^6\alpha - \frac{47}{6}\Lambda^9\alpha^2 - \frac{75}{2}\Lambda^{12}\alpha^3 - \frac{7437}{32}\Lambda^{15}\alpha^4 - \cdots \,, \\ W^{\rm gt, Sp(8)}_{\rm traceful} &= 5\Lambda^3 - 5\Lambda^6\alpha - 13\Lambda^9\alpha^2 - 65\Lambda^{12}\alpha^3 - \frac{2147}{5}\Lambda^{15}\alpha^4 - \cdots \,, \end{aligned} \qquad (4.13)$$

5. Comparison and discussion

According to the general conjecture, we are supposed to compare (3.9) with (4.8), and (3.6) with (4.13). We write $\triangle W \equiv W^{\rm DV} - W^{\rm gt}$, and find

$$\begin{aligned} \triangle W^{\rm Sp(4)}_{\rm traceless} &= 0 \cdot \Lambda^6\alpha + \Lambda^9\alpha^2 + 10\Lambda^{12}\alpha^3 + \cdots \,, \\ \triangle W^{\rm Sp(6)}_{\rm traceless} &= 0 \cdot \Lambda^6\alpha + 0 \cdot \Lambda^9\alpha^2 + \Lambda^{12}\alpha^3 + \cdots \,, \\ \triangle W^{\rm Sp(8)}_{\rm traceless} &= 0 \cdot \Lambda^6\alpha + 0 \cdot \Lambda^9\alpha^2 + 0 \cdot \Lambda^{12}\alpha^3 + \mathcal{O}\left(\Lambda^{15}\alpha^4\right) . \end{aligned} \qquad (5.1)$$

and

$$\begin{aligned} \triangle W^{\rm Sp(4)}_{\rm traceful} &= 0 \cdot \Lambda^6\alpha + \Lambda^9\alpha^2 + 20\Lambda^{12}\alpha^3 + \frac{32615}{81}\Lambda^{15}\alpha^4 + \cdots \,, \\ \triangle W^{\rm Sp(6)}_{\rm traceful} &= 0 \cdot \Lambda^6\alpha + 0 \cdot \Lambda^9\alpha^2 + \Lambda^{12}\alpha^3 + 30\Lambda^{15}\alpha^4 + \cdots \,, \\ \triangle W^{\rm Sp(8)}_{\rm traceful} &= 0 \cdot \Lambda^6\alpha + 0 \cdot \Lambda^9\alpha^2 + 0 \cdot \Lambda^{12}\alpha^3 + \Lambda^{15}\alpha^4 + \cdots \,. \end{aligned} \qquad (5.2)$$

We have indicated the terms that cancelled nontrivially by including them with a coefficient of zero. From these examples, we see that a disagreement sets in at order $(\Lambda^3)^h\alpha^{h-1}$, where $h = N/2 + 1$ is the dual Coxeter number. We also observe that the coefficient of the disagreement at this order is unity. We now discuss the implications of this result.

First, it is very unlikely that the discrepancy is due to a computational error, such as forgetting to include a diagram. This is apparent from the fact that the mismatch arises at a different order in perturbation theory for different rank gauge groups. So adding a new contribution to the Sp(4) result at order $\Lambda^9\alpha^2$, say, would generically destroy the agreement for Sp(6) and Sp(8) at this order. Instead, it is much more likely that our results indicate a breakdown of the underlying approach.

Let us return to the two basic elements of the DV conjecture. The first part asserts that the perturbative part of the glueball superpotential can be computed from matrix integrals, and the second part assumes that nonperturbative effects are captured by adding the Veneziano-Yankielowicz superpotential. We have proven the perturbative part of the conjecture for the relevant matter fields, but there are subtleties which we have so far avoided but now must discuss.

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In our perturbative computations we inserted no more than two $\mathcal{W}_\alpha$'s on any index loop, since we were interested in a superpotential for $S \sim \mathrm{Tr}\,\mathcal{W}^2$, and not in operators such as Tr $\mathcal{W}^{2n}$, $n > 1$. However, for a given gauge group, it may be possible to use Lie algebra identities to express such "unwanted" operators in terms of other operators, including S. Should we then include these new S terms along with our previous results?

This question might make one suspicious of the usual procedure, especially in our case given that the discrepancy sets in at order S^h, which is when we begin to find nontrivial relations involving S due to its underlying structure as a product of the fermionic field $\mathcal{W}_\alpha$. For example, for Sp(4) there are relations such as

$$\mathrm{Tr}\left[\left(\mathcal{W}^2\right)^3\right] = \frac{3}{4}\,\mathrm{Tr}\left[\mathcal{W}^2\right]\mathrm{Tr}\left[\left(\mathcal{W}^2\right)^2\right] - \frac{1}{8}\left(\mathrm{Tr}\left[\mathcal{W}^2\right]\right)^3. \tag{5.3}$$

So a naive guess is that the discrepancies can be accounted for if we keep all contributions coming from more than two $\mathcal{W}_\alpha$'s on an index loop, and re-express the traces of the form $\mathrm{Tr}(\mathcal{W}_\alpha)^{2n}$ $(n \geq h)$ in favor of S using relations like (5.3), setting all traces to zero that are not re-expressible in terms of S. Such considerations are indeed necessary in order to avoid getting unexpected results in certain cases, e.g. antisymmetric matter for Sp(2). Such a matter field is uncharged, and so one would expect (but see the discussion below) it to contribute a vanishing result for the glueball superpotential, but this is seen only if we compute all the trace structures. We should emphasize that if we keep all of these contributions then perturbation theory will not reduce to matrix integrals since the Schwinger parameter dependence will not cancel; nevertheless we can try this procedure and see what we get.

In order to check whether the above guess could be correct, we took a Φ^{2p} interaction and evaluated the perturbative superpotential explicitly, keeping all the traces. For this interaction, a discrepancy arises at the first order if we take $p > N$. Specifically, we considered a Φ^6 interaction in Sp(4) with antisymmetric matter. After a tedious calculation, we have found that this does *not* account for the discrepancy. In retrospect this is not really surprising, for two reasons. First, relations like (5.3) are corrected nonperturbatively. Second it is not obviously correct to set to zero terms like the one appearing in the middle of (5.3). One argument that we need not include such operators is that they have vanishing expectation values, so in the equations of motion they can be set to zero, leaving an equation of motion for S. However, this does not necessarily justify setting these operators to zero in the lagrangian before deriving the equations of motion for S. For example, when we encounter an expression like (5.3), we might have to replace it as

$$\mathrm{Tr}\left[\left(\mathcal{W}^2\right)^3\right] \to \mathcal{P}^{(3)}(S, \Lambda, \alpha) \tag{5.4}$$

where $\mathcal{P}^{(3)}$ is some polynomial in S of degree three that vanishes on shell, and include it in the effective superpotential.

Other nonperturbative subtleties may also play a role.[4] Even if the basic procedure is correct, discrepancies might seem to arise due to a redefinition of couplings and operators

[4]We thank Cumrun Vafa for bringing these issues to our attention.

involved in translating between gauge theory and matrix model expressions. It was argued in [17] that this is what happens in the $\mathcal{N} = 1^*$ theory. In that theory there were some constraints which could be imposed on possible operator redefinitions, but it is not clear whether this can also be done in our case. To check this one needs to consider all possible operator mixings, including mixing of single-trace and multi-trace operators.

It is also possible that the matrix model computation corresponds to a different gauge theory than the one we have been comparing with. In particular, starting from a string theory construction it is possible that there are some exotic nonperturbative effects which survive the field theory limit. In this case, matrix model results should be compared against theories in a different universality class than ordinary gauge theories. From this point of view it may be possible to justify nontrivial results for seemingly trivial theories, e.g. Sp(2) with traceless anti-symmetric matter. It would therefore be very instructive to find a string theory realization of our theories.

To summarize, our results indicate that for generic theories the simplest form of the DV conjecture is valid up to $(h-1)$ loops. On the other hand, the fact that our discrepancies arise in a very simple fashion — always with a coefficient of unity — suggests that perhaps there exists a way of modifying the DV recipe to enable us to go to h loops and beyond. Clearly, it is important to resolve these issues in order to determine the range of validity of the DV approach. One might hope that the approach will be useful for any $\mathcal{N} = 1$ theory admitting a large-N expansion. Our Sp(N) theories are certainly in this class, and so provide an important challenge.

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A. Diagrammatics for traceless matter field

In this appendix, we sketch the diagrammatics for evaluating the perturbative glueball superpotential, focusing on the case with a traceless tensor. To be specific, we consider the cubic interaction below:

$$e^{-W^{\mathrm{pert}}(S)} = \int \mathcal{D}\Phi\, e^{-\int d^4x\, d^2\theta\ \mathrm{Tr}\left[-\frac{1}{2}\Phi(\partial^2 - i\mathcal{W}^\alpha D_\alpha)\Phi + \frac{m}{2}\Phi^2 + \frac{g}{3}\Phi^3\right]}\,. \tag{A.1}$$

Namely, we consider SO with tracelss symmetric matter, or Sp with traceless antisymmetric matter.

At order g^2, there are four S^2 and RP^2 diagrams without disconnected propagators that contribute, as shown in figure 5 a)–d). These can be evaluated by combinatorics. For a) and b) there are 6 ways to contract legs. For c) and d), on the other hand, there are 3^2 ways to contract legs and 2 choices for the middle propagator (untwisted or twisted). Since there are two loop momenta, the glueball $S \sim \mathcal{W}^\alpha \mathcal{W}_\alpha$ should be inserted in two index loops, and we may insert only up to one glueball on each index loop. For S^2 graphs a) and c), there are 3 ways to do so, and a trace over the remaining index loop contributes N. For RP^2

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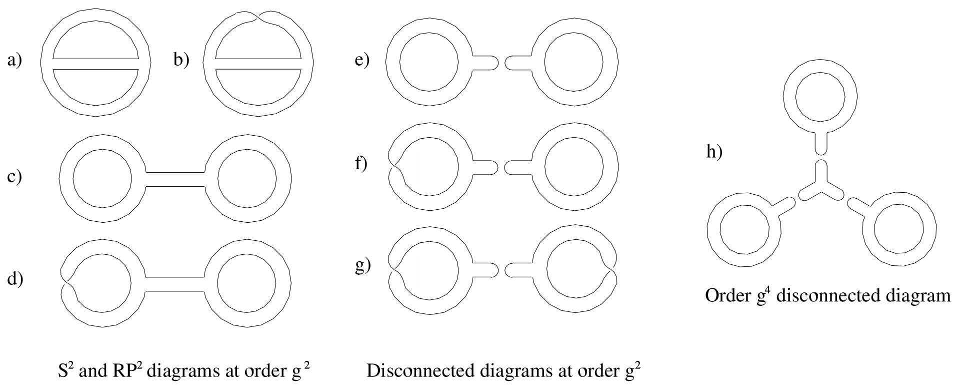

Figure 5: Diagrams for traceless tensor matter field.

graphs b) and d), there is only one way to insert the glueball, but the fermionic determinant gives an extra factor (± 4). The sign depends on the matter field under consideration see section 2.) Finally, for b) and d) there are respectively 3 and 2 ways to choose which propagator to twist. Therefore, the contributions are

$$f_{\rm a} = 6\cdot 3\cdot NS^2\,, \quad f_{\rm b} = 6\cdot 3\cdot(\pm 4)S^2\,, \quad f_{\rm c} = 3^2\cdot 2\cdot 3\cdot NS^2\,, \quad f_{\rm d} = 3^2\cdot 2\cdot 2\cdot(\pm 4)S^2\,. \tag{A.2}$$

Including factors coming from propagators and coefficients from Taylor expansion, we obtain

$$W^{(2)}_{\rm conn} = -\frac{g^2}{(2m)^3}\frac{1}{2!\,3^2}(f_{\rm a}+f_{\rm b}+f_{\rm c}+f_{\rm d}) = (-N\mp 3)\alpha S^2\,. \tag{A.3}$$

where we defined $\alpha = g^2/2m^3$ as before. This reproduces the first term of the traceful result (3.2).

For a traceless tensor, there are three additional diagrams e), f) and g), with disconnected propagators that give nonvanishing contributions.

These can be evaluated similarly to the connected ones. First, there are factors common to all three graphs; $(-2/N)$ from the disconnected propagator, and $3^2 = 9$ from the ways to contract legs. In addition, the particular graphs have the additional factors; e): $(2N)^2$ from the ways of inserting a glueball in one of two index loops in each S^2 component, and the trace on the remaining index loop. f): (± 4) from the fermionic determinant of the RP^2 component, and $2N$ from the glueball insertion into the S^2 component. Also, there is the same contribution from the $S^2\times RP^2$ graph. g): $(\pm 4)^2$ from two RP^2 components. Altogether we obtain

$$W^{(2)}_{\rm disconn} = -\frac{g^2}{(2m)^3}\frac{1}{2!\,3^2}\left(-\frac{2}{N}\right)\cdot 9\cdot\left[(2N)^2 + 2\cdot 2N\cdot(\pm 4) + (\pm 4)^2\right]S^2 = \frac{(2N\pm 4)^2}{4N}\alpha S^2\,. \tag{A.4}$$

Summing the connected and disconnected contributions, we obtain

$$W^{(2)}_{\rm conn} + W^{(2)}_{\rm disconn} = \left(\pm 1 + \frac{4}{N}\right)\alpha S^2 \tag{A.5}$$

which is the first term of the traceless result (3.7).

Higher order diagrams can be worked out in much the same way, although the number of diagrams increases rapidly. For the disconnected diagrams, we only have to consider the diagrams which are one-particle-reducible with respect to the disconnected propagator. Therefore, we basically just splice lower order diagrams with the disconnected propagator. The contribution is just the product of the contributions from the lower order pieces, multiplied by the ways to insert the disconnected propagator into them, and by $(-2/N)^n$ from the disconnected propagator itself. However, note that one should also consider diagrams such as h) of figure 5 In this case, the central D_2 piece contributes N from its index loop.

B. Summary of results from perturbation theory

In this appendix we state our results for $W^{\rm pert}(S,\alpha)$, the perturbative contributions to the glueball superpotential. These correspond to evaluating certain diagrams in the matrix model. We consider cubic interactions only,

$$W_{\rm tree} = \frac{m}{2}\,{\rm Tr}\;\Phi^2 + \frac{g}{3}\,{\rm Tr}\;\Phi^3\,, \tag{B.1}$$

which means that we will not consider the case of SO / Sp with adjoint matter. In any event, it is not necessary to compute the perturbative superpotential for the latter cases, since closed form expressions for even power interactions are already known [8, 9, 10]. The case of $\mathrm{U}(N)$ with adjoint matter is also well known [18], but for convenience we include it in the list below.

For traceful matter fields, instead of evaluating individual Feynman diagrams, there exists a much simpler method for computing which we have used to obtain the results below. We can simply compute the matrix model free energy by computer for certain low values of N. Since S^2 and RP^2 diagrams scale as N^{L+1} and N^L at L loops in perturbation theory, we can easily read off the S^2 and RP^2 contributions to any desired order. For traceless fields things are not so simple, since the N dependence becomes more complicated, and certain diagrams must be discarded (as discussed in section 2.)

We define

$$\alpha = \begin{cases} g^2/m^3 & \mathrm{U}(N) \\ g^2/2m^3 & \mathrm{SO}\,/\,\mathrm{Sp}(N). \end{cases} \tag{B.2}$$

$\mathrm{U}(N)$ **with adjoint matter.**

$$\begin{aligned} W^{\rm pert}(S,\alpha) &= N\frac{\partial F_{\chi=2}}{\partial S}\,, \qquad F_{\chi=2} = -\frac{S^2}{2}\sum_{k=1}^{\infty}\frac{(8\alpha S)^k}{(k+2)!}\frac{\Gamma\left(\frac{3k}{2}\right)}{\Gamma\left(\frac{k}{2}+1\right)} \\ W^{\rm pert}(S,\alpha) &= -2N\alpha S^2 - \frac{32}{3}N\alpha^2S^3 - \frac{280}{3}N\alpha^3S^4 - 1024N\alpha^4S^5 - \cdots\,. \end{aligned} \tag{B.3}$$

$\mathrm{SU}(N)$ **with adjoint matter.**

$$W^{\rm pert}(S,\alpha) = 0\cdot\alpha S^2 + 0\cdot\alpha^2S^3 + 0\cdot\alpha^3S^4 + \cdots\,. \tag{B.4}$$

$\mathrm{SO}(N)$ **with traceful symmetric matter.**

$$W^{\text{pert}}(S,\alpha) = -(N+3)\,\alpha S^2 - \left(\frac{16}{3}N + \frac{59}{3}\right)\alpha^2 S^3 - \\ - \left(\frac{140}{3}N + 197\right)\alpha^3 S^4 - \left(512N + \frac{4775}{2}\right)\alpha^4 S^5 - \cdots . \tag{B.5}$$

$\mathrm{SO}(N)$ **with traceless symmetric matter.**

$$W^{\text{pert}}(S,\alpha) = \left(1 + \frac{4}{N}\right)\alpha S^2 + \left(\frac{1}{3} - \frac{8}{N} - \frac{160}{3N^2}\right)\alpha^2 S^3 + \\ + \left(\frac{1}{3} - \frac{12}{N} + \frac{256}{3N^2} + \frac{3584}{3N^3}\right)\alpha^3 S^4 + \cdots . \tag{B.6}$$

$\mathrm{Sp}(N)$ **with traceful antisymmetric matter.**

$$W^{\text{pert}}(S,\alpha) = (-N+3)\,\alpha S^2 + \left(-\frac{16}{3}N + \frac{59}{3}\right)\alpha^2 S^3 + \\ + \left(-\frac{140}{3}N + 197\right)\alpha^3 S^4 + \left(-512N + \frac{4775}{2}\right)\alpha^4 S^5 + \cdots . \tag{B.7}$$

$\mathrm{Sp}(N)$ **with traceless antisymmetric matter**

$$W^{\text{pert}}(S,\alpha) = \left(-1 + \frac{4}{N}\right)\alpha S^2 + \left(-\frac{1}{3} - \frac{8}{N} + \frac{160}{3N^2}\right)\alpha^2 S^3 + \\ + \left(-\frac{1}{3} - \frac{12}{N} - \frac{256}{3N^2} + \frac{3584}{3N^3}\right)\alpha^3 S^4 + \cdots . \tag{B.8}$$

These results exhibit some remarkable cancellations. We find a vanishing result for $\mathrm{SU}(N)$ with adjoint matter, and a cancellation of the terms linear in N for $\mathrm{Sp}(N)$ with traceless antisymmetric matter. In both cases the cancellation seems to involve all the diagrams at a given order; for example, in figure 5, the $\mathcal{O}(N)$ contribution from e) cancels the $\mathcal{O}(N)$ contributions from both a) and c). Also, the cancellation does not seem to be special to cubic interactions; we have checked at leading order that the cancellation occurs for Φ^5 and Φ^7 interactions, even when both are present at the same time. Therefore the cancellation probably occurs for any tree level superpotential with odd power terms only. We do not have a proof of cancellation beyond the order indicated; it would be nice to provide one and to better understand the significance of this fact.[5]

C. Gauge theory results

In [14], a systematic method for determining the dynamical superpotential of the $\mathrm{Sp}(N)$ gauge theory with a traceless antisymmetric matter $\tilde{A}_{ab}$ was proposed. In this appendix, we briefly review the strategy, focusing on the $\mathrm{Sp}(8)$ case.

[5] This cancellation of the perturbative contribution to the effective superpotential is consistent with the more conventional gauge theory analysis of [19]. In fact, it is shown there that $W^{\text{eff}} = 0$ for any tree level superpotential with only odd power interactions.

First, we add to the theory $2N_F$ fundamentals Q_i. The moduli space of this enlarged $(N_{\tilde{A}}, N_F)$ theory is parameterized by

$$\tilde{O}_n = \mathrm{Tr}\left[(\tilde{A}J)^n\right], \qquad n = 2, 3, \ldots, \frac{N}{2} \tag{C.1}$$

as well as the antisymmetric matrices

$$\begin{aligned} M_{ij} &= Q_i^T J Q_j\,, \quad N_{ij} = Q_i^T J\tilde{A}JQ_j\,, \\ P_{ij} &= Q_i^T J(\tilde{A}J)^2 Q_j\,, \quad \ldots, \quad R_{ij} = Q_i^T J(\tilde{A}J)^{N/2-1}Q_j\,. \end{aligned} \tag{C.2}$$

The basic observation is that for $N_F = 3$, symmetry and holomorphy considerations restrict the dynamical superpotential to be of the form

$$W^{\mathrm{dyn}}_{(1,3)} = \frac{\text{Some polynomial in } \tilde{O}_n, M_{ij}, \ldots, R_{ij}}{\Lambda^{b_0}_{(1,3)}}\,, \tag{C.3}$$

where $b_0 = N - N_F + 4 = N + 1$ and the subscript $(1, 3)$ denotes the matter content $(N_{\tilde{A}}, N_F)$. The polynomial must of course respect the various symmetries of the theory. More significantly, the F-flatness equations following from $W^{\mathrm{dyn}}_{(1,3)}$ can be written in a Λ independent form. By setting $\Lambda = 0$, one sees that the equations must reduce to the classical constraints which follow upon expressing the gauge invariant field $\tilde{O}_n$, M_{ij}, …, R_{ij} in terms of their constituents Q_i and $\tilde{A}$. Quantum corrections to these classical constraints are forbidden by symmetry and holomorphy. These requirements fix $W^{\mathrm{dyn}}_{(1,3)}$ up to an overall normalization. Once we have obtained $W^{\mathrm{dyn}}_{(1,3)}$, we can derive the desired $W^{\mathrm{dyn}}_{(1,0)}$ by giving mass to Q_i and integrating them out.

For Sp(8), the above procedure uniquely determines the superpotential to be (this result appears in the second paper of [14])

$$\begin{aligned} W^{\mathrm{dyn}}_{(1,3)} = \frac{1}{\Lambda^9_{(1,3)}}\Big[&1152(PPP) + 6912(RPN) + 3456(RRM) - 864\tilde{O}_2(PNN) - \\ &- 1728\tilde{O}_2(RNM) + 108\tilde{O}_2^2(NNM) - 108\tilde{O}_2^2(PMM) + 9\tilde{O}_2^3(MMM) + \\ &+ 192\tilde{O}_3(NNN) - 576\tilde{O}_3(RMM) + 144\tilde{O}_2\tilde{O}_3(NMM) + \\ &+ 32\tilde{O}_3^2(MMM) + 432\tilde{O}_4(NNM) + 432\tilde{O}_4(PMM) - 36\tilde{O}_2\tilde{O}_4(MMM)\Big]\,, \end{aligned} \tag{C.4}$$

up to normalization, where $(ABC) \equiv \epsilon^{ijklmn}A_{ij}B_{kl}C_{mn}$. Now that we have obtained $W^{\mathrm{dyn}}_{(1,3)}$, we can integrate out Q_i by adding a mass term

$$W^{\mathrm{dyn}}_{(1,3)} \to W^{\mathrm{dyn}}_{(1,3)} + \frac{\mu^{ij}}{2}M_{ij}\,. \tag{C.5}$$

When solving the F-flatness condition, we can assume that $M_{ij}, N_{ij}, P_{ij}, R_{ij} \propto (\mu^{-1})_{ij}$ since μ^{ij} is the only quantity they can depend on. Plugging back in, we obtain the Sp(8) superpotential (4.5) and (4.6). The same procedure leads to the superpotential (4.3) and (4.4) for Sp(4) and Sp(6), respectively [14].

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